Modeling demand-price dependence in lot-sizing optimization

Jéssica Costa
Federal University of Rio de Janeiro (UFRJ)
Rio de Janeiro, RJ, Brazil
jessicacosta@cos.ufrj.br

Marcia Fampa
Federal University of Rio de Janeiro (UFRJ)
Rio de Janeiro, RJ, Brazil
fampa@cos.ufrj.br

Wendel Melo
Federal University of Uberlândia (UFU)
Uberlândia, MG, Brazil
wendelmelo@ufu.br

Fernanda Maria Pereira Raupp
National Laboratory for Scientific Computing (LNCC)
Petrópolis, RJ, Brazil
fernanda@lncc.br

ABSTRACT

We consider a convex mixed integer nonlinear program to model a lot-sizing optimization problem, in which demand decreases as the price of the sold product increases. To model the demand-price dependence, we propose a hyperbolic function and compare it to a linear demand-price dependence in the literature. By analyzing the parameters of the hyperbolic function, we show its potential to better model the realistic dependence between demand and price. We illustrate the solution of the problem by solving an instance with the Muriqui Optimizer solver and compare the hyperbolic and linear models.

KEYWORDS

lot-sizing, demand-price dependence, MINLP, convex relaxation

1 INTRODUCTION

Whenever certain operations are required between production runs of an item on a machine, lot-sizing appears naturally in production planning. In general, the setup cost of these operations does not depend on the quantity of products processed and, therefore, to minimize the costs, production must be run using large lot sizes. On the other hand, in a make-to-stock production system, this can generate high holding costs, as raw materials and some other components must be stored from the moment of production until the moment of shipment to the customer. Lot-sizing optimization aims to achieve the best trade-off between setup and holding costs, ensuring demand satisfaction and respecting the machine's production capacity.

Optimization models for production planning have been an important object of study for more than 5 decades. The seminal work of Wagner and Whitin [13] already addresses the formulation of the production planning problem for a single product in a single resource with unlimited production and inventory capacities. Since then, research on lot-sizing problems has focused on more realistic models and algorithmic approaches to enable their solution. Overviews in these topics are provided in [2, 5, 6]. A thorough study of mathematical formulations for the lot-sizing problems, seeking tight linear relaxations and good lower bounds for the problems, can be found in [11].

In this paper, we address the capacitated lot-sizing optimization problem that arises on a single level production planning of a single storable item. The classic version of this problem aims at the determination of the production over a planning horizon, which minimizes the total cost, given by the sum over all periods, of fixed setup costs and linear production and holding costs. A specific characteristic of this classic problem is that demand, which may vary over the time periods, is always met. Moreover, it is assumed that each positive production quantity during a period induces a setup, even in cases where production takes place in two consecutive periods.

We follow other works in the literature in an effort to adjust the assumptions of classic lot-sizing optimization models to more realistic situations, focusing on the dependence between demand and selling price. For that, instead of minimizing the total cost, we maximize gains, considering that revenue is given by a nonlinear function of demand in each period. Empirical evidences show that the classic assumption of a constant demand in each period of the planning horizon is an excessive simplification of reality. In [4], the authors relax this assumption considering a model where demand linearly decreases with the selling price. Other works consider different versions of lot-sizing problems with non-constant demands, addressing their dependence on inventory level [9, 10], price [1], price and marketing expenditure [12], and product's price and its environmental performance [15]. More realistic demand-price models are presented in [14], as well.

In [4], the revenue is modeled as a concave quadratic function of demand, leading to a convex Mixed Integer Quadratic Programming (MIQP) problem. The difficulty in solving this class of problems years ago is reflected in [4], which proposes a decomposition algorithm to solve the subproblem that arises when the integer variables of the model are fixed.

Taking advantage of the recent development in the area of Mixed Integer Nonlinear Programming (MINLP), we take a step forward in adjusting of the classic lot-sizing models to realistic situations, considering a more suitable nonlinear function to model the relationship between demand and price. We present a theoretical analysis of the function considered in our model to establish the correspondence between our nonlinear demand-price dependence and the linear dependence adopted in [4]. We show that our MINLP problem is also convex. Through a numerical example, we illustrate how the output for our model compares to the output for the MIQP model.

2 MATHEMATICAL FORMULATION

In this section, we present an MINLP formulation for the single-item capacitated lot-sizing problem, where the selling price at each period of the time horizon considered is dependent on the item demand. We will investigate the two following functions to model this dependence,

$$\begin{aligned} f^1(d_t) &:= \tfrac{1}{\beta_t}(\alpha_t - d_t), \\ f^2(d_t) &:= \tfrac{1}{\tau_t d_t + \mu_t} - \sigma_t, \end{aligned} \qquad (1)$$

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where $f^j(d_t)$ is the selling price value of the amount of demand d_t at period t, for each $j = 1, 2$, and $\alpha_t, \beta_t, \tau_t, \mu_t, \sigma_t$ are given parameters. From now on, we will distinguish between parameters and data. Parameters are associated with the definition of the functions in (1), as data refer to the instances data of the lot-sizing models.

The linear demand-price dependence formulated by f^1 was adopted in the single-item capacitated lot-sizing problem in [4]. The parameters $\alpha_t > 0$ and $\beta_t \in (0, 1)$ represent, respectively, the demand for zero price, and the decreasing rate of the demand as the price increases. The assumption of a decreasing demand as the price increases is certainly more realistic than the assumption of a given demand that does not depend on the price, and considering this in the lot-sizing problem leads to better decisions concerning both production and pricing. However, the constant decreasing rate of the demand as the price increase is still an approximation of reality, so we propose the alternative convex function f^2 to model the demand-price dependence.

Next, we consider this demand-price dependence in the standard formulation of lot-sizing optimization as a mixed integer program. Table 1 shows the notation used.

Data	
$t_{\max}$	number of periods in the time horizon T;
i_0	initial inventory level;
r_t	production capacity in period t;
h_t	unit holding cost at the end of period t;
c_t	unit production cost in period t;
q_t	setup cost in period t;
Variables	
d_t	demand in period t;
p_t	unit selling price in period t;
i_t	inventory level at the end of period t;
x_t	production quantity in period t;
y_t	binary variable (1, if there is production in period t);

Table 1: Notation

For each $j = 1, 2$, we formulate

$$(P_j)\ z := \max \sum_{t \in T} p_t d_t - c_t x_t - h_t i_t - q_t y_t \quad (2)$$

$$\text{subject to: } d_t = x_t + i_{t-1} - i_t, \quad t \in T \quad (3)$$

$$x_t \le r_t y_t, \quad t \in T \quad (4)$$

$$p_t = f^j(d_t), \quad t \in T \quad (5)$$

$$x_t, i_t, d_t, p_t \ge 0, \quad t \in T \quad (6)$$

$$y_t \in \{0, 1\}. \quad t \in T \quad (7)$$

The objective function (2) is the total gain given by the sum over all periods of the difference between the revenue and the total cost, i.e. production, holding and setup costs. Constraint (3) represents the flow balance constraint in period t, the inflows are the initial inventory i_{t-1} and the production x_t, the outflows are the demand d_t and the ending inventory i_t. Constraint (4) represents the capacity restriction and also fixes the setup variable y_t to 1 whenever there is positive production. Constraint (5) establishes how the demand depends on the price. Constraints (6-7) determine the domain of the variables. We note that, besides deciding on the lot size in each period, our problem model also decides on the selling price of the product.

Finally, in Proposition 1, we set the relationship between the parameters in the two models ($j = 1, 2$), and establish their range or sign. This will be relevant for the analysis presented in Section 3. In the remaining of the paper, the parameters used will follow the definitions in Proposition 1.

Proposition 1. *Let $\epsilon, \delta \in (0, 1)$. For each $t \in T$, let $\alpha_t > 0$, $\beta_t \in (0, 1)$, and define*

$$\sigma_t = \frac{\alpha_t}{\beta_t(1 + \frac{\delta}{1-\epsilon})^2}, \quad (8)$$

$$\mu_t := \beta_t / (\alpha_t + \beta_t \sigma_t), \quad (9)$$

$$\tau_t := \frac{1}{\delta \alpha_t}\Big(\sqrt{\frac{\mu_t}{\sigma_t}} - \mu_t\Big). \quad (10)$$

Then, all the parameters defined in (8)–(10) are positive.

Proof. The verification is trivial for σ_t and μ_t. For τ_t, we observe that

$$\frac{\mu_t}{\sigma_t} = \frac{\mu_t \beta_t (1 + \frac{\delta}{1-\epsilon})^2}{\alpha_t} > \frac{\mu_t \beta_t}{\alpha_t} > \frac{\mu_t \beta_t}{\alpha_t + \beta_t \sigma_t} = \mu_t^2.$$

□

3 AN ANALYSIS OF THE DEMAND-PRICE DEPENDENCE FUNCTIONS

The definition of the parameters in Proposition 1 establishes the correspondence between the two functions presented in (1) and the behavior of the two models investigated. In this section, we analyze similarities and differences among the models, and extract from this analysis how the assignment of values to the parameters can be used to better adapt them to different characteristics of real problems.

We start verifying in Proposition 2, that the demand becomes equal to zero when the price assumes the same value in all models, given by α_t / β_t.

Proposition 2. *We have $f^1(0) = f^2(0) = \frac{\alpha_t}{\beta_t}$.*

Proof. Clearly $f^1(0) = \frac{\alpha_t}{\beta_t}$. Also,

$$f^2(0) = \frac{1}{\mu_t} - \sigma_t := \frac{\alpha_t + \beta_t \sigma_t}{\beta_t} - \sigma_t = \frac{\alpha_t}{\beta_t}.$$

□

In Proposition 3, we see how the parameters ϵ and δ can also be used to determine the demand value for a zero price when considering f^2. We note that this demand value is α_t for f^1.

Proposition 3. *Let $\zeta := (1 + \frac{\delta}{1-\epsilon})^2$ and*

$$\hat{d}_t := \frac{\delta \zeta}{\sqrt{\zeta + 1} - 1} \alpha_t.$$

Then

$$f^2(\hat{d}_t) = 0.$$

Proof.

$$
\begin{aligned}
f^2(d_t) &= \frac{1}{\tau_t d_t + \mu_t} - \sigma_t \\
&= \frac{1}{\frac{1}{\delta\alpha_t}\left(\sqrt{\frac{\mu_t}{\sigma_t}} - \mu_t\right)d_t + \mu_t} - \sigma_t \\
&= \frac{1}{\frac{1}{\delta\alpha_t}\left(\sqrt{\frac{\beta_t}{(\alpha_t+\beta_t\frac{\alpha_t}{\zeta\beta_t})\frac{\alpha_t}{\zeta\beta_t}}} - \frac{\beta_t}{\alpha_t+\beta_t\frac{\alpha_t}{\zeta\beta_t}}\right)d_t + \frac{\beta_t}{\alpha_t+\beta_t\frac{\alpha_t}{\zeta\beta_t}}} - \frac{\alpha_t}{\zeta\beta_t} \\
&= \frac{\alpha_t}{\frac{1}{\delta}\left(\sqrt{\frac{\beta_t^2}{(\alpha_t^2(1+\frac{1}{\zeta}))\frac{1}{\zeta}}} - \frac{\beta_t}{\alpha_t(1+\frac{1}{\zeta})}\right)d_t + \frac{\beta_t}{1+\frac{1}{\zeta}}} - \frac{\alpha_t}{\zeta\beta_t} \\
&= \frac{\alpha_t}{\frac{\zeta\beta_t}{\delta\alpha_t}\left(\frac{\sqrt{\zeta+1}-1}{\zeta+1}\right)d_t + \frac{\zeta\beta_t}{\delta\alpha_t}\frac{\delta\alpha_t}{\zeta+1}} - \frac{\alpha_t}{\zeta\beta_t} \\
&= \frac{\alpha_t}{\zeta\beta_t}\frac{\delta(\zeta+1)\alpha_t}{(\sqrt{\zeta+1}-1)d_t + \delta\alpha_t} - \frac{\alpha_t}{\zeta\beta_t} \qquad (11)
\end{aligned}
$$

The result then follows by replacing d_t by $\hat{d}_t$ in (11). □

In the remaining of the section, we analyze the revenue function, given by the product between the item demand and the corresponding selling price defined by each function in (1). We verify that each revenue function has a unique maximum point, which varies with the parameters in α_t and δ.

Proposition 4. *Let $rev^j(d_t) := f^j(d_t) \cdot d_t$, for $j = 1, 2$. Then*

$$\frac{d}{d_t} rev^1(d_t) = \frac{1}{\beta_t}(\alpha_t - 2d_t), \qquad (12)$$

$$\frac{d}{dd_t} rev^2(d_t) = \frac{\mu_t}{(\tau_t d_t + \mu_t)^2} - \sigma_t, \qquad (13)$$

and

$$\frac{d^2}{d_t^2} rev^1(d_t) = -\frac{2}{\beta_t}, \qquad (14)$$

$$\frac{d^2}{dd_t^2} rev^2(d_t) = -2\frac{\mu_t \tau_t}{(\tau_t d_t + \mu_t)^3}. \qquad (15)$$

Corollary 1. *Let $rev^j(d_t) := f^j(d_t) \cdot d_t$, for $j = 1, 2$. We have*

- *$rev^1(d_t)$ is strictly concave for all $d_t \in \mathbb{R}$, and $d_t = \alpha_t/2$ is the unique critical point of $rev^1(d_t)$,*
- *$rev^2(d_t)$ is strictly concave for all $d_t \in \mathbb{R}$, and $d_t = \delta\alpha_t$ is the unique critical point of $rev^2(d_t)$.*

Proof. Considering Proposition 4, it is straightforward to verify all the results above, except the expression for the critical point of $rev^2(d_t)$, which comes from (10) and

$$\frac{d}{dd_t} rev^2(\delta\alpha_t) = \frac{\mu_t}{(\tau_t\delta\alpha_t + \mu_t)^2} - \sigma_t = \frac{\mu_t}{\left(\sqrt{\frac{\mu_t}{\sigma_t}} - \mu_t + \mu_t\right)^2} - \sigma_t = 0.$$

□

It is interesting to note that when using f^1, for each period t, the price becomes negative for demand values greater than α_t, therefore this is the maximum value for the d_t in the feasible set of the problem. Moreover, the maximum revenue always occurs when the demand reaches 50% of this maximum value. By varying the value of the parameter δ in f^2, on the other hand, we control the percentage of α_t that leads to the maximum revenue, gaining more flexibility to better model the decrease rate of demand as the price increases and also, the point where the maximum revenue occurs.

In Figure 1 we exemplify the plots of the two price and revenue functions, as well as the zeros of f^1 and f^2. Finally, we present the plots of the derivatives of the revenue functions highlighting the maximum revenue point, where the derivatives are equal to zero.

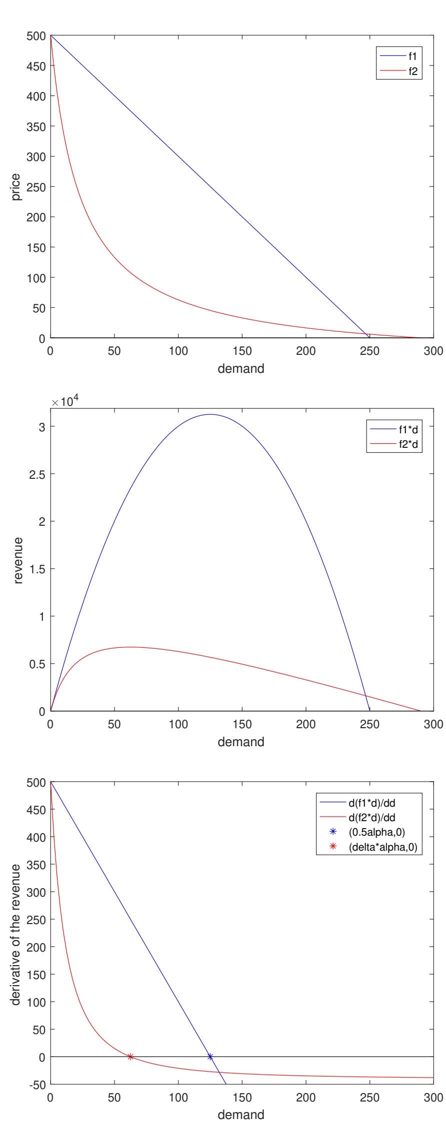

Figure 1: Modeling price vs. demand ($\alpha = 250, \beta = 0.5, \epsilon = 0.75, \delta = 0.4$)

From Corollary 1, we establish the following result, which allows us to apply algorithms for convex MINLP to (P_1) and (P_2), with guarantee of global optimality at the solution obtained.

Corollary 2. *Problems (P_j), for $j = 1, 2$, are convex MINLP problems, i.e., if (7) is replaced with $y_t \in [0, 1]$, for all $t \in T$, then the relaxation of (P_j) obtained is a convex optimization problem.*

4 A NUMERICAL EXAMPLE

In this section, we illustrate our modeling approaches with a small example. The parameters and the data for the two models are shown in Table 2.

$\epsilon := 0.25$; $\delta := 0.4$;
$\alpha := (104, 103, 92, 95, 91, 93)$;
$\beta := (0.19, 0.21, 0.36, 0.23, 0.38, 0.35)$;
$t_{\max} := 6$; $i_0 := 0$;
$r_t := (115, 157, 29, 101, 106, 77)$;
$h_t := (3.73, 1.56, 2.56, 1.09, 3.76, 3.55)$;
$c_t := (4.06, 1.62, 2.66, 1.14, 3.88, 3.50)$;
$q_t := (41.52, 15.53, 51.55, 3.69, 15.00, 8.37)$;

Table 2: Example (parameters and data)

In Tables 3–4, we present the numerical values of the variables at the optimal solution of the problem when considering both functions, f^1 and f^2, to model the demand-price dependence. In Figures 2–5, we show how the revenue and costs vary in the time horizon on the solution for f^1 and f^2, and in Figures 6–7 we compare the results over the entire time horizon.

The results shown were obtained in 0.03 second by Gurobi [3] for f^1, and in 1.25 second by Muriqui Optimizer [8] for f^2. The Hybrid Outer Approximation Branch-and-Bound (HOABB) algorithm from Muriqui Optimizer was applied (see [7] for details). We ran our experiments under Windows 10, on an Intel Core i7 processor, with 16GB RAM. We implemented our code in `Scilab`, `AMPL`, and `R`.

	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$	$t = 6$
p_t	277.64	246.35	129.94	207.56	120.43	135.07
d_t	51.25	51.27	45.22	47.26	45.24	45.73
x_t	51.25	96.49	0.00	92.50	0.00	45.73
i_t	0.00	45.22	0.00	45.24	0.00	0.00
y_t	1	1	0	1	0	1

Table 3: Solution of the example for f^1

	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$	$t = 6$
p_t	197.08	174.76	93.17	146.96	86.63	97.05
d_t	40.81	40.85	35.64	37.73	35.54	35.96
x_t	40.81	76.49	0.00	73.27	0.00	35.96
i_t	0.00	35.64	0.00	35.54	0.00	0.00
y_t	1	1	0	1	0	1

Table 4: Solution of the example for f^2

From the results in Tables 3–4 and from the plots in Figures 2–5, we see that for the parameters chosen, both models lead to a similar behavior for the optimal production along the time horizon. From the values of the variables y_t in the tables, and from the setup costs in Figure 5, we see that production occurs in the same periods for both models.

On the other hand, the faster decrease in the demand as the price increases, modeled by f^2, leads to a decrease in the revenues and also in the production and holding costs whenever they are positive, which we can observe in Figures 2–4. Furthermore, we see in the tables, that production decreases in all periods when f^2 models the demand-price dependence.

In Figures 6 and 7, by considering the entire time horizon, we see a clear decrease in production due to a more aggressive response of the demand to an increase in the price.

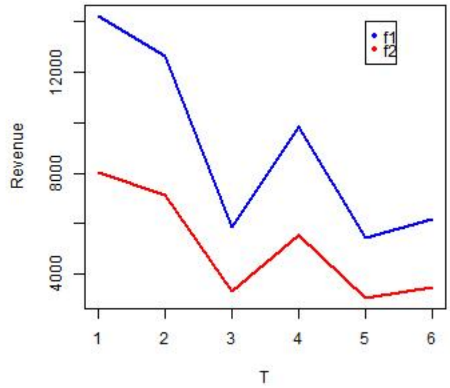

Figure 2: Revenue ($p_t d_t$)

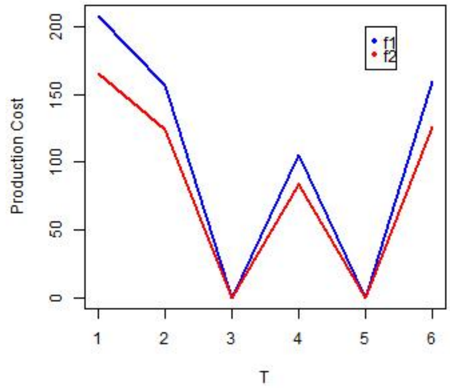

Figure 3: Production cost ($c_t x_t$)

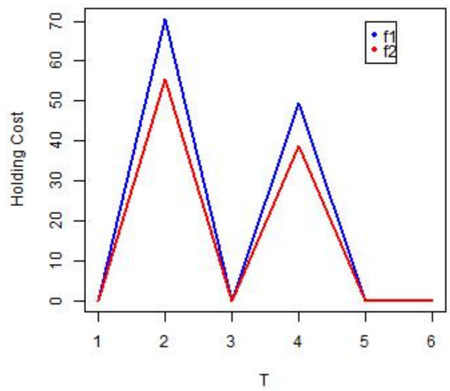

Figure 4: Holding cost ($h_t i_t$)

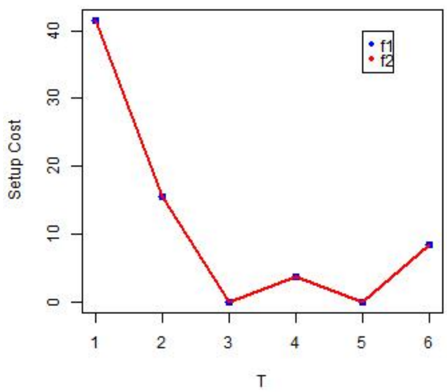

Figure 5: Setup cost ($q_t y_t$)

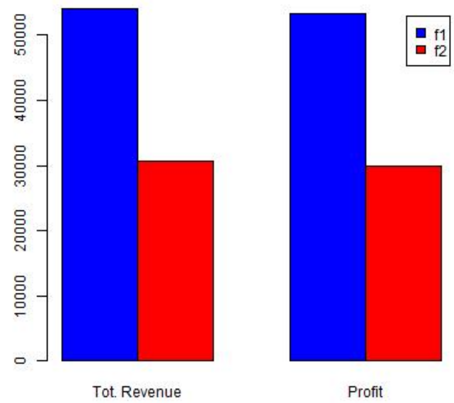

Figure 6: Total revenue and profit

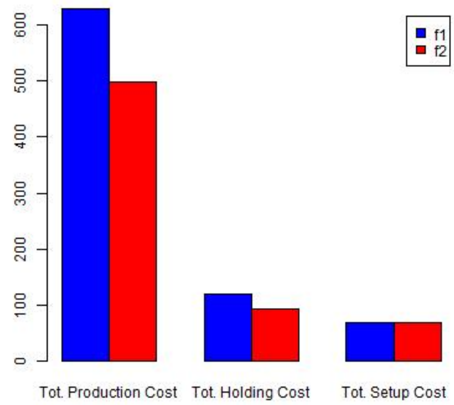

Figure 7: Total production, holding and setup costs

5 CONCLUSION

With the advance in the development of efficient algorithms and solvers for convex mixed integer nonlinear programming in the last decade, it becomes possible to better model production planning optimization problems. In this work, we exploit this possibility considering a hyperbolic function to model demand-price dependence in a lot-sizing problem. We discuss how the parameters in the function can be used to adjust the model to the characteristics of the application, and illustrate the application of the Muriqui optimizer solver to an instance of the problem formulated. Considering other functions to model demand-price dependence and a more comprehensive numerical experiment, we intend to demonstrate in future work, how the advance in convex MINLP can be useful to solve more realistic lot-sizing problems, leading to better decisions concerning production.

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