Short Paper

New polynomial circular-arc instances of tuple domination in graphs

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ABSTRACT

It is well-known that 1-tuple domination (classical domination) is NP-hard for general graphs. For circular-arc graphs, it is efficiently solvable due to M.S. Chang (1998). On the other side, efficient algorithms of a known generalization —k-tuple domination (k fixed)— of 1-tuple domination are not developed for circular-arc graphs and k greater than 2. In this work we introduce a new circular-arc graph subclass. For this subclass, we present a lower bound for the k-tuple domination number for every positive integer k. Finally, we find the exact value of these numbers by proving how to achieve this bound.

KEYWORDS

web graph, circular-arc graph, domination, vertex set partition

1 INTRODUCTION

Domination in graphs is useful in different applications. There exist many variations — such as k-tuple domination among others— regarding slight differences in their definitions. These differences make circular-arc graph subclasses adequate and useful mostly due to their relation to "circular" issues, such as in forming sets of representatives, in resource allocation in distributed computing systems, in coding theory [11], and in testing for circular arrangements of genetic molecules [6].

The decision problem (fixed k) associated with k-tuple domination is NP-hard [7] but polynomial time solvable in some graph classes (see for example [1, 3, 7, 8]). For proper interval graphs, efficient algorithms for this problem are developed in [2] for $k = 1$, and in [7] for the remaining values of k (in fact this algorithm was built for strongly chordal graph which constitute a superclass of proper interval graphs).

In the class of circular-arc graphs (superclass of proper interval graphs), it follows from previous works by Bui-Xuan et al. (2013) and by Belmonte et al. (2013) —in the context of locally checkable vertex subset problems in graph classes with quickly computable and bounded min-width— that the k-tuple domination problem is solvable in time $O(|V(G)|^{6k+4})$. More efficient algorithms for this class are presented for 1-tuple domination in [2], and for 2-tuple domination in [9].

It remains challenging to find algorithms for these problems in subclasses of circular-arc graphs for values of k greater than 2 that are more efficient than the existing ones.

In [4], the authors give a faster (linear) algorithm for this problem on web graphs (which are regular circular-arc graphs with certain symmetry).

Our aim is to continue finding faster algorithms for k-tuple domination in subclasses of circular-arc graphs. In this direction we introduce a subclass of circular-arc graphs that generalizes some web graphs. We use the results in [4] to find the exact value of the k-tuple domination number as well as a minimum k-tuple dominating set for this new graph class, for all positive integer number k.

This contribution is organized as follows. In Section 2 we present some basic definitions and results that we use throughout the paper. In Section 3 we introduce the definition of Q-web graphs and present some of their properties. Sections 4 and 5 are devoted to study the k-tuple domination number of Q-web graphs for all values of k.

2 PRELIMINARIES

We consider finite simple graphs, where $V(G)$ and $E(G)$ denote the vertex and edge sets respectively, of a graph G. When the graph is clear from the context, we simply write V and E to respectively denote $V(G)$ and $E(G)$.

Given a graph G and $S \subseteq V$, the *subgraph induced by* S is denoted by $G[S]$. When $G' = G[S]$ for some $S \subseteq V$, G' is an *induced subgraph* of G. The graph $G - S$ stands for $G[V \setminus S]$ and, for simplicity, we write $G - v$ instead of $G - \{v\}$, for $v \in V$. A graph G' is an *edge subgraph* of G if $V(G') = V(G)$ and $E(G')$ is a subset of $E(G)$.

The *complete graph* K_r is the graph having r vertices all of which are pairwise adjacent.

A *clique* in G is a subset $Q \subseteq V$ inducing a complete graph in G.

An *independent set* in G is a subset of pairwise nonadjacent vertices in G.

The *closed neighborhood* of $v \in V$ is $N[v] = N(v) \cup \{v\}$, where $N(v) = \{u \in V(G) : uv \in E\}$. When we need to mention the graph G explicitly, we write $N_G(v)$ (or $N_G[v]$). The *degree* of vertex $v \in V$ is $\deg(v) = |N(v)|$.

The *minimum degree* of G, denoted by $\delta(G)$, is the minimum cardinality of $N(v)$ between all vertices $v \in V$.

A graph G is *circular-arc* if it has an intersection model consisting of arcs in a circle, i.e. if there is a one-to-one correspondence between the vertices of G and a family of arcs on a circle such that two distinct vertices are adjacent in G when the corresponding arcs intersect in the circle.

Given positive integer numbers n and m with $m \geq 1$ and $n \geq 2m + 1$, the *web* graph W_n^m is the graph having vertex set $\{v_1, \ldots, v_n\}$ and $v_i v_j$ being an edge in $E(W_n^m)$ if $j \equiv i \pm l$, $(mod\ n)$,

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$l \in \{1, \ldots, m\}$ [10]. From its definition, it is clear that $\deg(v) = 2m$ for each $v \in V(W_n^m)$.

Given a graph G, with $V = \{v_1, \ldots, v_n\}$, its *adjacency matrix* is the square $n \times n$ 0, 1-matrix $N(G) = (m_{ij})$, where $m_{ij} = 1$ if and only if $v_i v_j \in E(G)$. Note that $N(G)$ is symmetric and has 0's on the main diagonal. The *augmented adjacency matrix* or *neighborhood matrix*, $N[G]$, is defined as $N[G] = N(G) + I$, where I is the identity matrix of appropriate size. For a graph G, the *characteristic vector* of a set $R \subseteq V$ is the vector on $|V|$ components, whose component i is 1 when $i \in R$ and is 0 when $i \notin R$. If we index the row set of $N[G]$ by $V = \{v_1, \ldots, v_n\}$, the i-th row of $N[G]$ is the characteristic vector of $N[v_i]$, for $i = 1, \ldots, n$.

According to A. Tucker [11], a 0, 1-matrix satisfies the *circular 1's property* (Circ1P) for columns if its rows can be permuted so that the l's in each column are circular (appear in a circularly consecutive fashion by thinking of the matrix as wrapped around a cylinder). It is not difficult to see that the augmented adjacency matrix of a web graph satisfies the Circ1P for columns. It is proved in [11] that graphs whose augmented adjacency matrix has the Circ1P for columns are circular-arc.

Given G and a non-negative integer k, $D \subseteq V$ is a *k-tuple dominating set* in G if in every closed neighbourhood there are at least k elements of D; i.e. $|N[v] \cap D| \geq k$, for each $v \in V$. Notice that G has a k-tuple dominating set if and only if $k \leq \delta(G) + 1$. Also, if G has a k-tuple dominating set D, then $|D| \geq k$. When $k \leq \delta(G) + 1$, the size of a k-tuple dominating set of minimum size in G is denoted by $\gamma_{\times k}(G)$ and called *k-tuple domination number* of G [5]. If $k > \delta(G)+1$, $\gamma_{\times k}(G)$ is defined as ∞. Observe that $\gamma_{\times 1}(G) = \gamma(G)$, the classical domination number, i.e. the concept of tuple domination generalizes the well-known concept of domination in graphs. Besides, note that $\gamma_{\times 0}(G) = 0$ for every graph G. When G is not connected, the k-tuple domination number of G is the sum of the k-tuple domination numbers of its connected components. Throughout this work G is a connected graph and k, an integer number with $k \leq \delta(G)+1$. Given a graph G and a fixed positive integer k, the *k-tuple domination problem* is to find a k-tuple dominating set in G of size $\gamma_{\times k}(G)$.

3 Q-WEB GRAPHS

In this section we introduce a new subclass —Q-web graphs— of circular-arc graphs and give the first lower bound for the k-tuple domination number in Q-web graphs.

Definition 3.1. For non negative integer numbers m, s, j with $m, s \geq 3$ and $1 \leq j \leq m-1$, the *Q-web graph* $Q(j, s, m)$ is defined as follows. All sums in the subscripts are taken modulo s.

The vertex set of $Q(j, s, m)$ has sm elements and is partitioned into the $2s$ sets of the form:

- $A_i = \{(i-1)(m-j)+1, \ldots, (i-1)(m-j)+m-j\}$, for $i = 1, \ldots, s$ and
- $P_i = \{1^i, \ldots, j^i\}$, for $i = 1, \ldots, s$.

The edge set of $Q(j, s, m)$ is defined as follows:

- the subgraph induced by $\bigcup_{i=1}^{s} A_i$ is the web graph $W_{s(m-j)}^{m-j}$;
- P_i is a clique, for every $i = 1, \ldots, s$;
- for $i = 1, \ldots, s$, the adjacencies between the vertices in P_i and P_{i+1} are given by $l^i h^{i+1}$ for $h = 1, \ldots, l-1$ and $l = 2, \ldots, j$;
- every vertex in P_i is adjacent to every vertex in $A_i \cup A_{i-1}$ for $i = 1, \ldots, s$.

From the definition above, observe that $\bigcup_{t=1}^{s}\{l^t\}$ is an independent set in $Q(j, s, m)$, for each $l = 1, \ldots, j$;

Example 3.2. According to Definition 3.1, the vertex set of $Q(2, 5, 5)$ is partitioned into $A_1 = \{1, 2, 3\}$, $A_2 = \{4, 5, 6\}$, $A_3 = \{7, 8, 9\}$, $A_4 = \{10, 11, 12\}$, $A_5 = \{13, 14, 15\}$, $P_1 = \{1^1, 2^1\}$, $P_2 = \{1^2, 2^2\}$, $P_3 = \{1^3, 2^3\}$, $P_4 = \{1^4, 2^4\}$ and $P_5 = \{1^5, 2^5\}$.

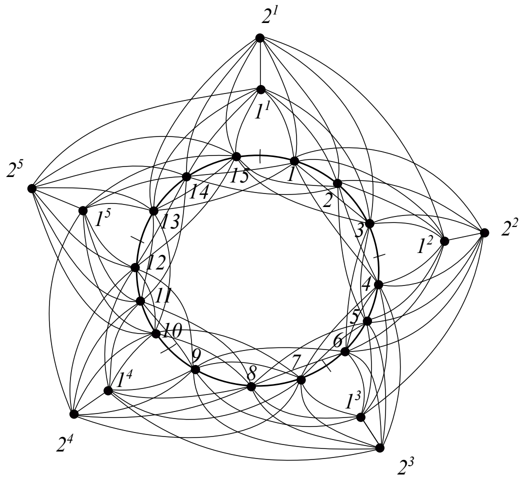

Figure 1: The graph $Q(2, 5, 5)$ in Example 3.2.

For $i = 1, \ldots, s$, P_i is called the *i-th leg* and, for $p \in \{1, \ldots, j\}$ $L_p = \bigcup_{i=1}^{s}\{p^i\}$ is called the *p-th level set* of $Q(j, s, m)$. It is not difficult to check that $\deg(v) = 2(m-1)$ for each $v \in P_i$ and $i = 1, \ldots, s$ and $\deg(v) = 2m$ for $v \in \bigcup_{i=1}^{s} A_i$.

When necessary, the subsets A_i, P_i and L_i of V are indicated respectively as $A_i(j, s, m)$, $P_i(j, s, m)$ and $L_i(j, s, m)$.

The following relationship will be crucial and states that every Q-web graph is an induced subgraph of other Q-web graph. More precisely, for every j, s and m it holds:

$$Q(j, s, m) = Q(j+1, s, m+1) - L_{j+1}(j+1, s, m+1).$$

Figure 2 shows an scheme of graphs $Q(j, s, m)$ for $j = 1, 2, 3$ and fixed s and m. For the sake of clarity, we have not included all the edges having one endpoint in P_i and the other in A_i but only one of them, and the same for those with both endpoints in $\bigcup_{i=1}^{s} A_i$.

Let us now prove that the neighbourhood matrix of a Q-web graph has the Circ1P for columns. Thus, due to Tucker [11], it turns out that Q-web graphs are circular-arc.

Proposition 3.3. *If G is a Q-web graph then $N[G]$ has the Circ1P for columns.*

Proof. Let $G = Q(j.s, m)$, for some $m, s \geq 3$ and j with $1 \leq j \leq m-1$. Consider matrix $N[G]$ whose rows are indexed as follows, for each $i = 1, \ldots, s$:

- for $t \in \{(i-1)m+1, (i-1)m+2, \ldots, (i-1)m+j\}$, the t-th row of $N[G]$ is the characteristic vector of the closed neighborhood of vertex $(t-(i-1)m)^i$ of P_i;
- for $t \in \{ij+(i-1)(m-j)+1, ij+(i-1)(m-j)+2, \ldots, ij+(i-1)(m-j)+(m-j)\}$, the t-th row of $N[G]$ is the characteristic vector of the closed neighborhood of vertex $t-ij$ of A_i.

This proves that $N[G]$ has the Cic1P for columns. □

The first graph of Figure 3 shows the graph $Q(1, 3, 5)$, where the dotted lines correspond to a 3-cycle that were deleted from the web graph W_{15}^5. In general, the graph $Q(j, s, m)$ is an edge

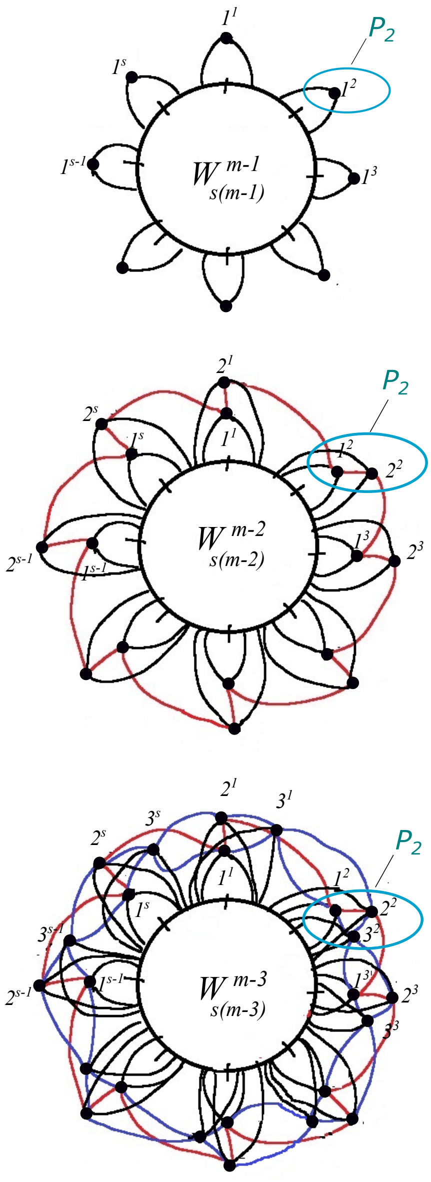

Figure 2: Scheme of Q-web graphs $Q(j, s, m)$, for $j = 1, 2, 3$.

subgraph of the web W_{sm}^{m} that is obtained after deleting from W_{sm}^{m}, the edges in j consecutive cycles of length s.

We observe that the definition of Q-web graphs could be extended by considering $j = 0$ also, meaning that "no cycle is deleted'. However, since the k-tuple domination problem is already solved for web graphs, we keep the definition of Q-web graphs for $j \geq 1$.

Since $\delta(Q(j, s, m)) = 2m - 2$ for all j, s and m, in the sequel k is always a non negative integer with value at most $2m - 1$.

Let $Q(j, s, m)$ be a Q-web graph for some j, s and m and D be a k-tuple dominating set in $Q(j, s, m)$. From the fact that $Q(j, s, m)$ is an edge subgraph of the web W_{sm}^{m}, it is not difficult to check that D is a k-tuple dominating set of the web graph W_{sm}^{m} as well. Due to the fact that the k-tuple domination number of webs is known [4], the following natural lower bound arises, for every $k \leq 2m - 1$:

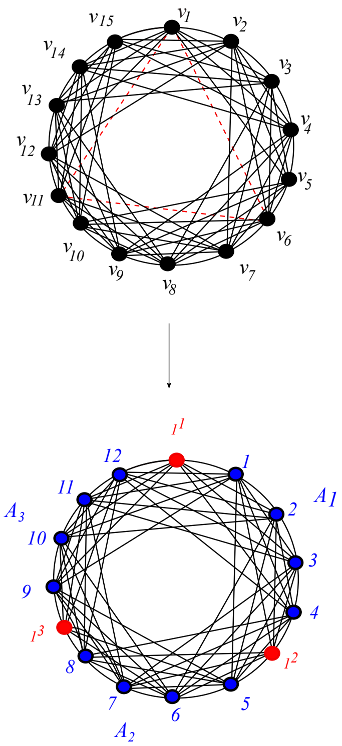

Figure 3: $Q(1, 3, 5)$ as an edge subgraph of W_{15}^{5}. Relabelling of $Q(1, 3, 5)$ (vertices in red are those in L_1).

$$\gamma_{\times k}\left(Q(j, s, m)\right) \geq \gamma_{\times k}\left(W_{sm}^{m}\right) = \left\lceil \frac{ksm}{2m+1} \right\rceil. \tag{1}$$

4 TUPLE DOMINATION ON GRAPHS $Q(1, s, m)$

Let us begin this section by improving, for $j = 1$, the lower bound in Equation (1).

Proposition 4.1. *Let $Q(1, s, m)$ be a Q-web graph with $s, m \geq 3$ and k be any non negative integer with $k \leq 2m - 1$. Then, it holds that*

$$\gamma_{\times k}\left(Q(1, s, m)\right) \geq \left\lceil \frac{ks}{2} \right\rceil.$$

Proof. Let D be a k-tuple dominating set in $Q(1, s, m)$ and consider the partition of $V(Q(1, s, m))$ given by the sets P_i and A_i with $i = 1, \ldots, s$.

Let $t := |D \cap L_1|$. Then we have

$$2\,|D| = \sum_{i=1}^{s} |A_i \cap D| + \sum_{i=1}^{s} |A_{i+1} \cap D| + 2t. \tag{2}$$

From their definition, $N(1^i) = A_i \cup A_{i-1}$ for each $i = 1, \ldots, s$ (sum taken modulo s). Since D is a k-tuple dominating set in $Q(1, s, m)$ and using (2), we have

$$\sum_{i=1}^{s} k \leq \sum_{i=1}^{s} \left|N[1^i] \cap D\right| = t + \sum_{i=1}^{s} |A_i \cap D| + |A_{i-1} \cap D| = 2|D| - t,$$

and thus

$$|D| \geq \frac{ks + t}{2} \geq \frac{ks}{2}.$$

Since $\gamma_{\times k}(G) \in \mathbb{Z}^{+}$ for every graph G, we have the desired bound. □

We prove in the next theorem that the lower bound in Proposition 4.1 can be achieved with $j = 1$. Then we obtain the exact value of the k-tuple domination number of $Q(j,s,m)$ for $j = 1$ that will lead in the following section to compute the corresponding domination number for every Q-web graph.

Theorem 4.2. *Let $Q(1,s,m)$ be a Q-web graph with $s, m \geq 3$ and k any non negative integer with $k \leq 2m - 1$. Then, it holds that*

$$\gamma_{\times k}(Q(1,s,m)) = \begin{cases} \dfrac{ks}{2} & \text{for even } k, \\[2ex] \dfrac{(k-1)s}{2} + \left\lceil \dfrac{s}{2} \right\rceil & \text{for odd } k. \end{cases}$$

Proof. Following Proposition 4.1, it only remains to prove that there exists a k-tuple dominating set D in $Q(1,s,m)$ of the desired size.

Suppose k is even, i.e. $k = 2r$ for some $r \geq 1$. We define the set $D = \bigcup_{j=1}^{s} A'_j$ where, for each $j = 1,\dots,s$, $A'_j = \{i_1^j, \dots, i_r^j\}$ is a subset of r consecutive vertices of A_j and such that

$$|i_r^j - i_1^{j+1}| = m - 1 - r.$$

We refer as A_jA_{j+1}*-vertices*, to the $m-1-r$ consecutive vertices $l_{j_1}, l_{j_2}, \dots, l_{j_{m-1-r}}$ in the set $(A_j \cup A_{j+1}) \setminus (A'_j \cup A'_{j+1})$ (see Figure 4). By definition, it holds that

$$i_1^j < i_2^j < \cdots < i_r^j < l_{j_1} < \cdots < l_{j_{m-1-r}} < i_1^{j+1} < \cdots < i_r^{j+1}.$$

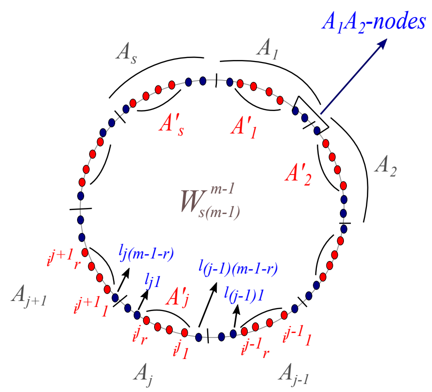

Figure 4: Notation for the proof of Theorem 4.2.

Let $v \in V(Q(1,s,m))$. We divide our analysis according to $v \in L_1$ or not, and prove in each case, that v has at least $k = 2r$ neighbours in the set D.

If $v = 1^i \in L_1$ then $|N[1^i] \cap D| = |A'_{i-1} \cup A'_i| = k$, for each $i = 1,\dots,s$. Thus $|N[v] \cap D| \geq k$.

If $v \in V(Q(1,s,m)) \setminus L_1$ —i.e. $v \in A_{j^*}$ for some $j^* \in \{1,\dots,s\}$— since $A'_{j^*} \subseteq A_{j^*}$ we have $A'_{j^*} \subseteq N[v]$, and then

$$|N[v] \cap A'_{j^*}| = |A'_{j^*}| = r. \tag{3}$$

We then show that v has r additional neighbours in $D \setminus A'_{j^*}$. We divide the proof into the two possible cases: $v \notin D$ or $v \in D$.

If $v \notin D$ and v is a $A_{j^*}A_{j^*+1}$-vertex, since $v \in A_{j^*}$ we have

$$|v - i_r^{j^*+1}| \leq |l_{j^*_1} - i_r^{j^*+1}| = |l_{j^*_1} - l_{j^*_{(m-1-r)}}| + |l_{j^*_{(m-1-r)}} - i_r^{j^*+1}| = \\ = m - 1 - r - 1 + r = m - 2.$$

This implies $A'_{j^*+1} \subset N[v]$ and then $|N[v] \cap A'_{j^*+1}| = r$. Otherwise ($v \notin D$ and v is a $A_{j^*-1}A_{j^*}$-vertex), since $v \in A_{j^*}$ we have

$$|v - i_1^{j^*-1}| \leq |l_{(j^*-1)_{m-1-r}} - i_1^{j^*-1}| = m - 1 - r - 1 + r = m - 2.$$

This implies $A'_{j^*-1} \subset N[v]$ and then $|N[v] \cap A'_{j^*-1}| = r$. In both cases, it holds $|N[v] \cap D| \geq r + r = k$.

If $v \in D$, i.e. $v = i_t^j$ for some $t \in \{1,\dots,r\}$, we consider w such that $|i_t^j - i_w^{j+1}| = m-1$. Since $|A'_j| = r$, it holds $r-t+1+m-1-r+w = m-1$ implying $w = t-1$ (see Figure 5). Analogously, let p satisfy $|i_t^j - i_p^{j-1}| = m-1$. Then $t-1+m-1-r+r-p+1 = m-1$, implying $p = t$.

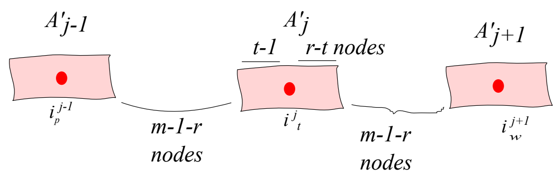

Figure 5: $v = i_t^j$ for some $t \in \{1,\dots,r\}$ in the proof of Theorem 4.2

Then $|N[i_t^j] \cap (A'_{j-1} \cup A'_{j+1})| = w+r-p+1 = t-1+r-t+1 = r$. Again we conclude that $|N[v] \cap D| \geq k$.

In all, D is a k-tuple dominating set in $Q(1,s,m)$ in this case.

When k is odd, let us define the set $D := \bigcup_{j=1}^{s} A'_j$, where $A'_j = \{i_1^j, \dots, i_{r+1}^j\}$ for odd j and $A'_j = \{i_1^j, \dots, i_r^j\}$ for even j, are in each case consecutive vertices of $W_{s(m-1)}^{m-1}$ and $|i_{r+1}^j - i_1^{j+1}| = m - 1 - |A'_j|$. We follow a similar reasoning as the one when k is even, by splitting it into the subcases given by odd s and even s. In both cases we conclude that D is a k-tuple dominating set in $Q(1,s,m)$ of the desired size. □

5 TUPLE DOMINATION ON GRAPHS $Q(j,s,m)$, $j > 1$

We are able to solve the problem for $Q(j,s,m)$ for $j > 1$. Recall that

$$Q(j-1,s,m-1) = Q(j,s,m) - L_j(j,s,m)$$

for every j, s and m, where $L_j(j,s,m)$ is the j-th level of $Q(j,s,m)$.

Theorem 5.1. *Let $Q(j,s,m)$ be a Q-web graph with $s, m \geq 3$, $1 \leq j \leq m-1$ and k be any non negative integer with $k \leq 2m-3$. Then, it holds that*

$$\gamma_{\times k}(Q(j,s,m)) = \begin{cases} \dfrac{ks}{2} & \text{for even } k, \\[2ex] \dfrac{(k-1)s}{2} + \left\lceil \dfrac{s}{2} \right\rceil & \text{for odd } k. \end{cases}$$

Sketch of the Proof. We prove by induction on j for $2 \leq j \leq m-1$, that

$$\gamma_{\times k}(Q(j,s,m)) = \gamma_{\times k}(Q(j-1,s,m-1)),$$

and then apply the result of Theorem 4.2.

The result holds for $j = 2$ by noting that $Q(2,s,m)$ is isomorphic to an edge subgraph of $Q(1,s,m)$ and then clearly, given a k-tuple dominating set in $Q(2,s,m)$, this set is a k-tuple dominating set in $Q(1,s,m)$ as well. Thus,

$$\gamma_{\times k}(Q(1,s,m)) \leq \gamma_{\times k}(Q(2,s,m)).$$

Since $k \leq 2m-3 = \delta(Q(1,s,m-1))+1$, and the k tuple domination number does not depend on m from Theorem 4.2, we have

$$\gamma_{\times k}(Q(1,s,m)) = \gamma_{\times k}(Q(1,s,m-1)).$$

On the other hand, from the proof of Theorem 4.2, there exists a minimum k-tuple dominating set D in $Q(1,s,m-1)$ such that

$$D \subseteq \bigcup_{i=1}^{s} A_i(1,s,m-1).$$

It can be checked that $|N_{Q(2,s,m)}[v] \cap D| \geq k$ for every $v \in V(Q(2,s,m))$, thus D is a k-tuple dominating set in $Q(2,s,m)$ and

$$\gamma_{\times k}(Q(2,s,m)) \leq |D| = \gamma_{\times k}(Q(1,s,m-1)).$$

In all

$$\gamma_{\times k}(Q(2,s,m)) = \gamma_{\times k}(Q(1,s,m-1)),$$

and in this way we have proved the base case.

The proof of the inductive step follows a similar reasoning. □

Up to now we have solved the k-tuple domination problem in Q-web graphs $Q(j,s,m)$ except for $k = 2m-2$ and $k = 2m-1$. Since these two values of k require a different treatment, we present them separately.

Theorem 5.2. *Let $Q(j,s,m)$ be a Q-web graph with $s,m \geq 3$. Then, it holds that*

$$\gamma_{\times(2m-2)}(Q(j,s,m)) = sm - \left\lfloor \frac{s}{2} \right\rfloor.$$

Sketch of the Proof. For the proof we consider the set

$$D = \bigcup_{i=1}^{s} A_i \cup \bigcup_{p=2}^{j} L_p \cup D_1$$

where $D_1 \subseteq L_1$, $D_1 \cap P_1 \neq \emptyset$, if $D_1 \cap P_i \neq \emptyset$ then $D_1 \cap P_{i+1} = \emptyset$ for all $i = 1,\dots,s$, and such that $|D_1| = \frac{s}{2}$ if s is even and $|D_1| = \frac{s-1}{2} + 1$ if s is odd, and prove that D is a k-tuple dominating set in $Q(j,s,m)$. In this way we have

$$\gamma_{\times(2m-2)}(Q(j,s,m)) \leq |D| = s(m-2) + s + \left\lceil \frac{s}{2} \right\rceil = sm - \left\lfloor \frac{s}{2} \right\rfloor.$$

We then prove that, given a k-tuple dominating set D in $Q(j,s,m)$ and a set $S \subseteq V \setminus D$ with $|S| = \left\lfloor \frac{s}{2} \right\rfloor + 1$, then nor S is a subset of L_p for $p = 1,\dots,j$ neither S is a subset of $\bigcup_{i=1}^{s} A_i$, leading to

$$|D| \geq sm - \left\lfloor \frac{s}{2} \right\rfloor.$$

The result follows. □

Theorem 5.3. *Let $Q(j,s,m)$ be a Q-web graph with $s,m \geq 3$ and $1 \leq j \leq m-1$. Then, it holds that*

$$\gamma_{\times(2m-1)}(Q(j,s,m)) = sm.$$

Proof. We only have to note that, for $t \in \{1,\dots,j\}$, $\deg(t^i) = 2m-2$ for each $i = 1,\dots,s$. This implies that every $(2m-1)$-tuple dominating set of $Q(j.s.m)$ must include the union of all the neighborhoods $N[t^i]$ for all $i = 1,\dots,s$ and $t \in \{1,\dots,j\}$. Since this union is the whole vertex set $V(Q(j,s,m))$, the result follows. □

We apply the results of Theorems 5.1, 5.2 and 5.3 to obtain the k-tuple domination numbers for the graph $Q(2,5,5)$ in Example 3.2 (see Figure 1).

Example 5.4. For the graph in Example 3.2 we have:

$$\gamma_{\times k}(Q(2,5,5)) = \begin{cases} 3 & \text{if } k=1, \\ 5 & \text{if } k=2, \\ 8 & \text{if } k=3, \\ 10 & \text{if } k=4, \\ 13 & \text{if } k=5, \\ 15 & \text{if } k=6, \\ 18 & \text{if } k=7, \\ 23 & \text{if } k=8, \\ 25 & \text{if } k=9. \end{cases}$$

Corollary 5.5. *Given integer numbers m,s,j with $m,s \geq 3$ and $1 \leq j \leq m-1$ and the vertex set partition of of $Q(j,s,m)$ given by A_i and P_i, for $i = 1,\dots,s$, the k-tuple domination problem can be solved efficiently on $Q(j,s,m)$ for any fixed k.*

6 CONCLUSIONS

As far as we know, the most recent results concerning k-tuple domination in graph subclasses of circular-arc graphs were given for web graphs in [4]. In this work we introduced the class of Q-web graphs that generalizes certain web graphs, and proved that they have the Circ1P for columns, thus being circular-arc graphs.

The main contribution of this paper is to find faster algorithms that do not depend on k, for the k-tuple domination problem on the new subclass of circular-arc graphs introduced. As a by-product, a k-tuple dominating set on Q-web graphs can be obtained explicitly.

The results obtained up to now inspire to continue the study of k-tuple domination in circular-arc graphs, in order to find efficient algorithms that do not depend on k in the whole graph class.

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